Propagation of error

Generalized propagation of error

For the general case of a function that depends on multiple variables, f(x,y,z), we Can expand the function in a Taylor's series

$$f(x,y,z) \approx f^0 + \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial z}z$$

To see that the function depends in a linear fashion on each variable with a slope equal to the first derivative of the function with respect to that variable. The combined error with respect to all of the variables is root-mean-square average

$$\sigma = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \Delta y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \Delta z^2 + \cdots}\,.$$

Propagation of error

If we calculate a quantity Q from a number of measured values (A, B, C,..) each with their respective uncertainties (σ(A), σ(B), σ(C),..) we can calculate how the uncertainties *propagate* into the value of Q as follows

If Q = f(x,y,z) then

$$\sigma^2(Q) \cong \left(\frac{\partial f}{\partial x}\right)^2 \sigma^2(x) + \left(\frac{\partial f}{\partial y}\right)^2 \sigma^2(y) + \left(\frac{\partial f}{\partial z}\right)^2 \sigma^2(z)$$

Please note that we take derivatives versus '**x, y, z etc**' here. Then we evaluate the derivative at the average or best fit value for a given parameter.

This formula is an *approximation* that only holds true if the error sources σ(z),σ(y),σ(z), .. are independent (i.e. uncorrelated). If x and y represent e.g. the intercept and slope from the same regression this is generally not true. In addition we tacitly assume that we can replace the σ's by their estimates (i.e. s_e's)

Propagation of error

Most commonly, the error on a quantity, Δx , is given as the standard deviation, σ
The standard deviation is the positive square root of variance, σ^2

The value of a quantity and its error are often expressed as an interval $x \pm \Delta x$

If the statistical probability distribution of the variable is known, it is possible to derive confidence limits to describe the region within which the true value of the variable may be found.

In the case of a line $y = mx + b$

We can calculate the error of y relative x as $\Delta y = m\,\Delta x$

Propagation of error in a polynomial

For example if $Q = f(x) = x^n$.

$$\frac{\partial f}{\partial x} = nx^{n-1}$$

$$\sigma^2(Q) = (nx^{n-1})^2\sigma^2(x)$$

$$\sigma(Q) = \frac{\sigma(x)nx^n}{x}$$

$$\frac{\sigma(Q)}{Q} = n\frac{\sigma(x)}{x}$$

From propagation of error we can conclude that the relative error in Q is n times larger than the relative error in x if the relationship is $Q = x^n$.