Analysis of a coin flip

Stochastic variables

Calculating probability

Normal distribution

Robust statistics

Flipping a coin

We know intuitively that the probability of getting heads on a single flip is P =1/2. On two flips the probably of getting heads twice is P = (1/2)(1/2) = 1/4. We can generalize this and say the probably of getting n heads in a row is P = $(1/2)^n$. We can call the total number of possible combinations of heads/tails for a given number n, the combinatoric.

$$C = 2^n$$

The binomial distribution

Suppose we ask what the probability is of getting heads 14 times out of 20.
We can imagine 20 bins and in each bin the coin is either heads or tails.
In that case the number of ways that this can occur is given by the binomial distribution.

$$W = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where n is the total number of flips and k is the number of heads. So if we flip the coin 20 times and get 14 heads this can happen many was (T = tail, H = head).

TTTTTTHHHHHHHHHHHHHH
TTTTTHTHHHHHHHHHHHHH
TTTTHHTTHHHHHHHHHHHH
…

The total number of ways that that heads and tails can appear is given by

$$W_{14\ heads} = \binom{20}{14} = \frac{20!}{14!\,6!} = 38760$$

Discrete probabilities of 14 heads out of 20 flips

Putting these together we can calculate the probability of obtaining 14 heads on 20 flips.

$$W_{14\ heads} = \binom{20}{14} = \frac{20!}{14!\,6!} = 38760$$

The probability of getting 14 heads is this value divided by the combinatoric for 20 flips.

$$C = 2^{20} = 1{,}048{,}675$$

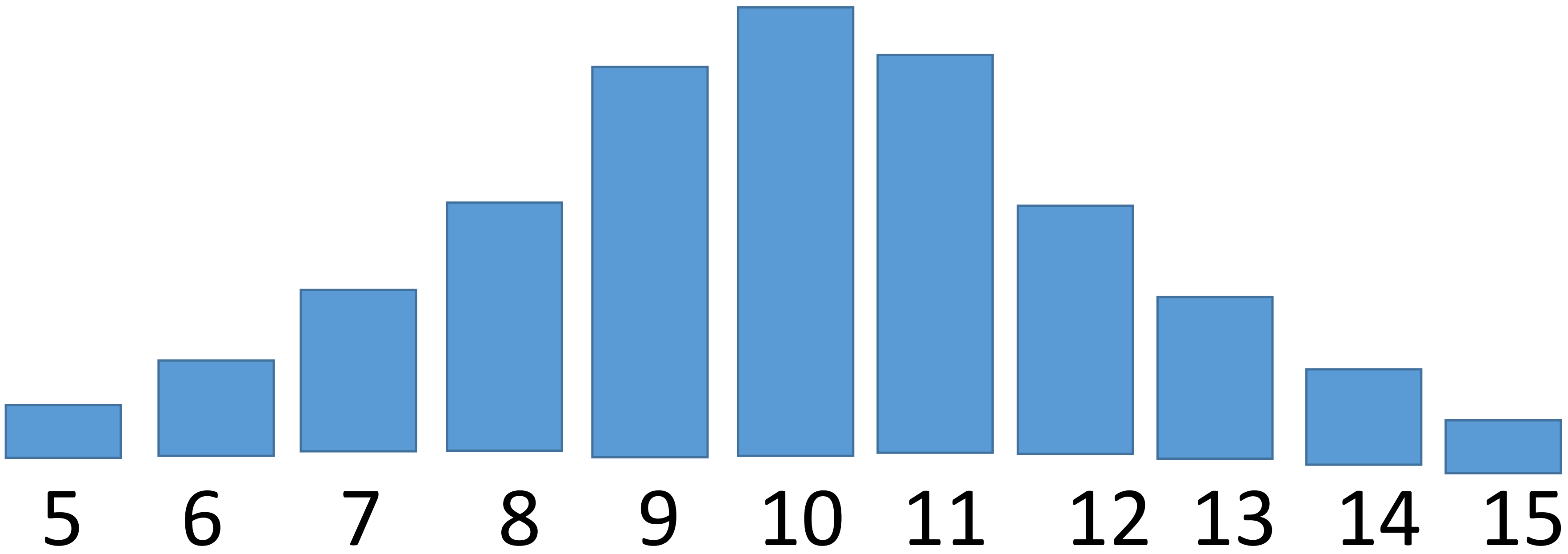
So,

$$P = \frac{W}{C} = \frac{1}{2^{20}}\binom{20}{14} = \frac{38{,}760}{1{,}048{,}675}$$
$$= 0.0369$$

or 3.69%.

What is the distribution of discrete probabilities?

What would it look like if we plotted the probability as
A function of the number of heads from 0 to 20.
We know that the maximum will be 10 (if the coin is fair).
On either side the probability will decrease.



An example: testing a coin for fairness

Let's assume that a coin is not loaded, i.e. it is a fair coin. Then the null hypothesis would say that the coin should have an equal probability for heads and tails. If we flip the coin 20 times, we would predict that the coin should be heads on 10 of those flips (based on probability).

Trial 1. Flip the coin 20 times and find that there are 11 heads and 9 tails. Here we use the binomial distribution to figure out the probability. The analysis on the next slide shows that the probability is 16%. If we assume therefore that $p = 0.16$, then we see that $p > 0.05$ and we have no basis to reject the null hypothesis. We accept the null hypothesis that the coin is fair.

The total number of ways that that heads and tails can appear is given by

$$W_{11\ heads} = \binom{20}{11} = \frac{20!}{11!\,9!} = 167960$$

The probability of getting 11 heads is this value divided by the combinatoric for 20 flips.

$$C = 2^{20} = 1{,}048{,}675$$

So,

$$P = \frac{W}{C} = \frac{1}{2^{20}}\binom{20}{11} = \frac{167{,}960}{1{,}048{,}675} = 0.160$$

or 16%.

Example: testing a coin for fairness

Trial 2. Flip the coin 20 times and find that there are 17 heads and 3 tails. The binomial distribution gives us

$$W_{17\ heads} = \binom{20}{17} = \frac{20!}{17!\,3!} = 1140$$

The probability of getting 17 heads is this value divided by the combinatoric for 20 flips.

$$C = 2^{20} = 1{,}048{,}675$$

So,

$$P = \frac{W}{C} = \frac{1}{2^{20}}\binom{20}{17} = \frac{1140}{1{,}048{,}675}$$

$$= 0.0010$$

In this instance $p < 0.001$, which is a significant deviation from the Mean of the null hypothesis. Therefore, the null hypothesis is invalid and the coin is not fair.