Analysis and treatment of errors in the adiabatic compression laboratory experiment

Theory

The lab manual asks you to derive the relationship solving for C_p. I did it for C_v instead, but obviously if we know C_v, we know C_p. The error in the two is the same.

$$\gamma = \frac{C_p}{C_v} = \frac{C_v + R}{C_v}$$

$$\gamma C_v = C_v + R$$

$$(\gamma - 1)C_v = R$$

$$C_v = \frac{R}{(\gamma - 1)}$$

Experimental data

Based on the adiabatic expansion formula, we have

$$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$$

where,

$$\gamma = \frac{C_p}{C_v}$$

In general the PV^{γ} product here is equal to a constant

$$C = PV^{\gamma}$$

which leads to

$$\ln(C) = \ln(P) + \gamma \ln(V)$$

or referenced to an initial pressure the equation can be set up for a linear regression.

$$\ln\left(\frac{P_i}{P_0}\right) = -\gamma \ln\left(\frac{V_i}{V_0}\right)$$

By this point once have completed the measurement you will also have an error in the volume based on the fit to the standard line.

Error analysis

There are two ways to treat this.

1. You could determine the error in γ from the fit to the linearized data. Then C_v would related as above

$$C_v = \frac{R}{(\gamma - 1)}$$

using the propagation of error.

$$\sigma(C_v) = \sqrt{\left(\frac{\partial C_v}{\partial \gamma}\right)^2 \sigma_\gamma^2 + \left(\frac{\partial C_v}{\partial V}\right)^2 \sigma_V^2}$$

Where both σ_γ and σ_V are known to you from your measurements of the two calibration lines.

$$\frac{\partial C_v}{\partial \gamma} = -\frac{R}{(\gamma - 1)^2}$$

$$\frac{\partial C_v}{\partial V} = \frac{\partial C_v}{\partial \gamma}\frac{\partial \gamma}{\partial V}$$

Since

$$\gamma = -\frac{\ln\left(\frac{P}{P_0}\right)}{\ln\left(\frac{V}{V_0}\right)}$$

From above we have

$$\frac{\partial \gamma}{\partial V} = \frac{\ln\left(\frac{P}{P_0}\right)}{\ln\left(\frac{V}{V_0}\right)^2}\frac{1}{V} = -\frac{\gamma}{\mathrm{V}\ln\left(\frac{V}{V_0}\right)}$$

Where we can use the mid-point volume of the range used in the determination in order to obtain a numerical value for the derivative.

$$\sigma(C_v) = \frac{R\gamma}{(\gamma-1)^2}\sqrt{\left(\frac{\sigma_\gamma}{\gamma}\right)^2 + \frac{1}{\ln\left(\frac{V}{V_0}\right)^2}\left(\frac{\sigma_V}{\mathrm{V}}\right)^2}$$

$$\frac{\sigma(C_v)}{C_v} = \frac{\gamma}{(\gamma-1)}\sqrt{\left(\frac{\sigma_\gamma}{\gamma}\right)^2 + \frac{1}{\ln\left(\frac{V}{V_0}\right)^2}\left(\frac{\sigma_V}{\mathrm{V}}\right)^2}$$

2. Alternatively, treat γ as a function of P and V. Note that the error in P can be obtained either from the estimate of the manufacturer PASCO or from propagation of error from the relationship of P and V in the ideal gas law (given in the laboratory manual).

$$\gamma(P,V) = -\frac{\ln\left(\frac{P}{P_0}\right)}{\ln\left(\frac{V}{V_0}\right)}$$

$$\sigma(\gamma) = \sqrt{\left(\frac{\partial\gamma}{\partial P}\right)^2\sigma_P^2 + \left(\frac{\partial\gamma}{\partial V}\right)^2\sigma_V^2}$$

$$\frac{\partial\gamma}{\partial V} = \frac{\ln\left(\frac{P}{P_0}\right)}{\ln\left(\frac{V}{V_0}\right)^2}\frac{1}{V} = -\frac{\gamma}{\mathrm{V}\ln\left(\frac{V}{V_0}\right)}$$

$$\frac{\partial\gamma}{\partial P} = -\frac{\frac{1}{P}}{\ln\left(\frac{V}{V_0}\right)} = -\frac{\gamma}{\mathrm{P}\ln\left(\frac{P}{P_0}\right)}$$

$$\frac{\sigma(\gamma)}{\gamma} = \frac{\ln\left(\frac{V}{V_0}\right)}{\ln\left(\frac{P}{P_0}\right)}\sqrt{\frac{1}{\ln\left(\frac{P}{P_0}\right)^2}\left(\frac{\sigma_P}{P}\right)^2 + \frac{1}{\ln\left(\frac{V}{V_0}\right)^2}\left(\frac{\sigma_V}{\mathrm{V}}\right)^2}$$

$$\frac{\sigma(\gamma)}{\gamma} = \frac{1}{\ln\left(\frac{P}{P_0}\right)}\sqrt{\frac{1}{\gamma^2}\left(\frac{\sigma_P}{P}\right)^2 + \left(\frac{\sigma_V}{\mathrm{V}}\right)^2}$$

Hint: you should be able to write all quantities in terms of their relative error in the final expression, e.g.

$$\left(\frac{\sigma_P}{P}\right), \qquad \left(\frac{\sigma_V}{V}\right), \qquad \left(\frac{\sigma_\gamma}{\gamma}\right)$$

The point of these formulae is that a small error in one quantity, e.g. the volume may end up contributing to a larger error in another quantity say γ because of the functional dependence of one on the other. To see the functional dependence always write it out,

$$\gamma(P, V)$$

Or

$$C_v(\gamma)$$

Bottom line: The final error estimate you should use is the largest of the possible ways to estimate the error. You may be able to take a short cut that underestimates your error, but a careful approach should take all of the sources of error into account.