

## Tutorial 3. Matrices and Complex Numbers

Actually both matrices and complex numbers become a lot more interesting (and fun) in a spreadsheet than they are on a blackboard or in a math test. And yes they are quite useful in quantitative science. In future labs we will see some more applications and you may need them in your project phase.

### Rotation

- Start a new sheet and put the following data in A1:B5

1	0
2	0
3	0
4	0
5	0
- Type in C1: `=COMPLEX(A1,B1)` (This creates a complex number  $A1+B1.i$ )
- Type in D1: `=IMPRODUCT(C1,$G$1)` (*Hint: Use F4 to put in the \$ signs*). This function multiplies two complex numbers.
- Type in E1: `=IMREAL(D1)`. (Guess what this does?)
- Type in F1: `=IMAGINARY(D1)`. (Guess what this does?)
- Select C1:F1 and double click the bottom right corner of F1 to fill down to F5
- Type in G1: `=IMEXP(COMPLEX(0,H3))` (More about this in a minute..)
- Type in H3: `=PI()*I3`
- Type in I3: `=0.25`
- Select A1:B5 and make a scatter plot with only symbols
- Select E1:F5 and make a scatter plot with only symbols
- While the second chart is active, copy it using Ctrl+C
- Activate the first chart and *paste* using Ctrl+V
- Delete the second chart
- In the first chart change the axis range to -6 to +6 fixed values for both X and Y and stretch to make the graph look square.
- Press Alt+F11. This takes you to the VBA IDE
- On the left you should see all the VBA projects currently open, including the workbook you are working in. Click that project to activate and go to the Insert menu. Insert a *module*
- In the middle pane a new window should open. Type the following program in it

```
Sub rotate()  
a = Range("I3").Value  
a = a + 0.1  
Range("I3").Value = a  
End Sub
```

- Either press Alt+F11 again or click on the small Excel icon top left to go back to your workbook
- Click your chart to activate and right click.
- On the popup assign the *rotate* module to the chart
- Click on an arbitrary cell, then on the chart and again and again.

### **Roots of one**

The above demonstration makes use of a set of complex numbers with very special properties. You could call them the  $n^{\text{th}}$  roots of unity, because they are the (complex) solutions of the equation:

$$x^n = 1$$

There is always  $n$  such roots. These are known as the "roots of unity", which is the keyword you should use if you want to read more about these numbers in Wikipedia.

Question: What are these roots for  $n=2$ ? For  $n=4$ ? To answer this question we should introduce the unit circle of radius 1 in the complex plane. The complex plane has a real axis ( $x$ -axis) and an imaginary axis ( $y$ -axis).

The *absolute value* or *magnitude* of the  $m^{\text{th}}$  root:  $|x_m| = \sqrt{x_m * x_m}$  of these numbers is always one and that means that in the complex plane they lie on the *unit circle* because the radius represents the magnitude  $|x|$ . In complex exponential notation the roots are easy to write:

$$: x_m = \exp 2\pi i [m/n] \quad \text{where } m = 1, 2, 3, 4, \dots, n$$

It is this function that we used to make things rotate in the demonstration above, because *multiplying* with such a number corresponds to a *rotation* in the complex plane.

According to the *Euler rule* for complex exponentials,

$$e^{\pm ix} = \cos(x) \pm i \sin(x)$$

the real part of our root-of-one is  $\cos 2\pi [m/n]$  and the imaginary part  $\sin 2\pi [m/n]$ .

To find the roots we will also need the De Moivre formula:

$$(\cos(x) \pm i \sin(x))^n = \cos(nx) \pm i \sin(nx)$$

Let's have look at these numbers using an Excel spreadsheet.

1. Go to a new sheet
2. Type in A1: 0; type in A2: 1;
3. Select the two cells, put the cursor on the bottom right corner of the selected range until it changes into a +. Then drag down to A15. (This should *fill* the range with the numbers  $m$ .)
4. In B1 type: `=A1*2*PI()/15`; (This gives  $2\pi [m/n]$  with  $n=15$ )
5. In C1 type: `=IMEXP(COMPLEX(0,B1))` (giving  $\exp 2\pi i [m/n]$ ; `COMPLEX(a,b)` gives:  $a+b.i$ )
6. In D1 type: `=IMREAL(c1)` (The Re part  $\cos 2\pi [m/n]$ )
7. In E1 type: `=IMAGINARY(c1)` (The Im part  $\sin 2\pi [m/n]$ )

8. Now select the range B1:E1, put the cursor on the bottom right corner of E until a + appears and double click
9. The formulas should now have filled down to row 15. In the B column we have calculated  $2\pi[m/n]$ . This value is also known as the **phase angle**.
10. In the C column we have  $\exp(2\pi i[m/n])$  and in the D and E columns we have its real and imaginary parts  $\cos(2\pi[m/n])$  and  $\sin(2\pi[m/n])$
11. Now select the range D1:E1 and make a scatter-plot with only markers, no lines. Provided you stretch the chart a little it should look like a perfect circle. (Make it so!)
12. In A17 type `=CORREL(E1:E15,D1:D15)` These are the roots of  $x^{15}=1$ . Look at the angles between them. How big are those and why are they so regularly spaced? What is the meaning of the contents of A17? Are the numbers random?
13. In C16 type `=IMPRODUCT($C$3,C1)`. This calculates the product of the two complex roots in C1 and C3. Drag the contents down to C30 to fill. Select d15:e15 and double click + to fill down to E30. Make a second graph of D15:E30.
14. The graphs look identical but if you look at the numbers in rows 15 to 30 you'll see there is a difference. (What?). Activate the second graph and press Ctrl+c to copy, then click on the first graph and press Ctrl+v to paste.
15. Change the formula in C16 to `=IMPRODUCT(COMPLEX(0,1),C1)` and double click the right bottom corner + to fill.  
The function `complex(0,1)` represents the complex number  $0+1i = i$ . Was this number one of the original roots? What happens if you multiply by it?  
In next week's lab we will see an important application of these numbers the Fourier Transform.

Let's do rotation in a different way by using a **rotation matrix**.

- Go back to your first sheet
- Type in H5: `=cos(H3)`
- Type in I6: `=cos(H3)`
- Type in I5: `=-sin(H3)`
- Type in H6: `=cos(H3)`
- Select the range E1:F1, type the array function: `=MMULT(A1:B1,$H$5:$I$6)` (*hint: you can type `=MMULT(` and then use the mouse to select the A1:B1 range, then type a comma, then use the mouse to select the H5:I6 range, then use F4 to put in the \$\$ signs). Activate this array function with Ctrl+Shift+Enter*
- We have now multiplied our (x,y) coordinates with our rotation matrix. Use the bottom right + double click to copy the new function over all five rows
- Click on the graph to see what happens

As you see 2D rotations can be done both ways: complex or matrix. In 3D it gets more complicated. There are no 3D complex numbers, but there are 4D ones, known as quaternions and they do the trick in 3D. Most computer games use quaternions. Euler matrices in 3D also exist but have a mathematical problem known as 'gimbal lock' which makes your program crash.

## Using Matrix Commands in Excel

### General Advice and Background

Matrices are used in many practical problems. For example, when you produce a diagram, most of the time, matrices appear in the diagram. More importantly, many calculations are most easily performed by using matrices. In this class, we will learn to enter and to perform various operations of matrices in Excel. We will use these skills to some practical problems in the next class. It is important that you know the basic properties of matrices in order to deal with them in Excel.

### Entering and Naming Matrices

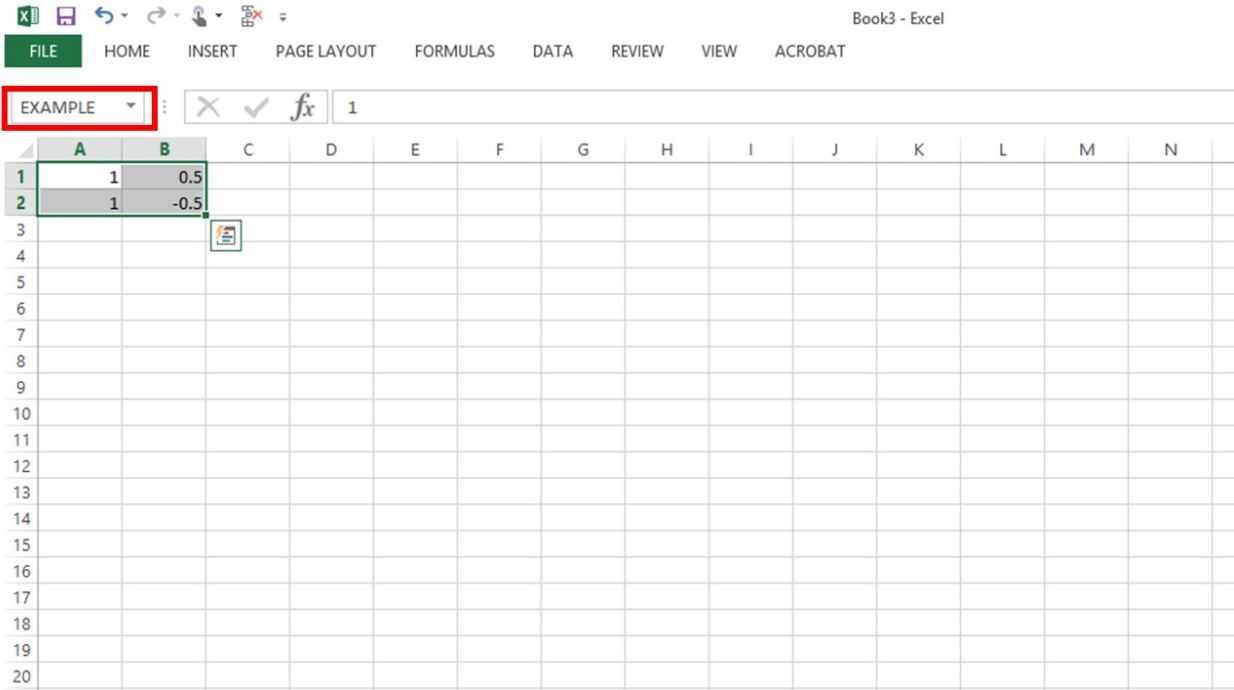
Before entering the elements of any matrix, you should always enter the **Matrix Name**, such as P, Q, etc. You may also want to enter the **Description** of the matrix, like Coefficient Matrix, Product of P and Q, etc.

Let

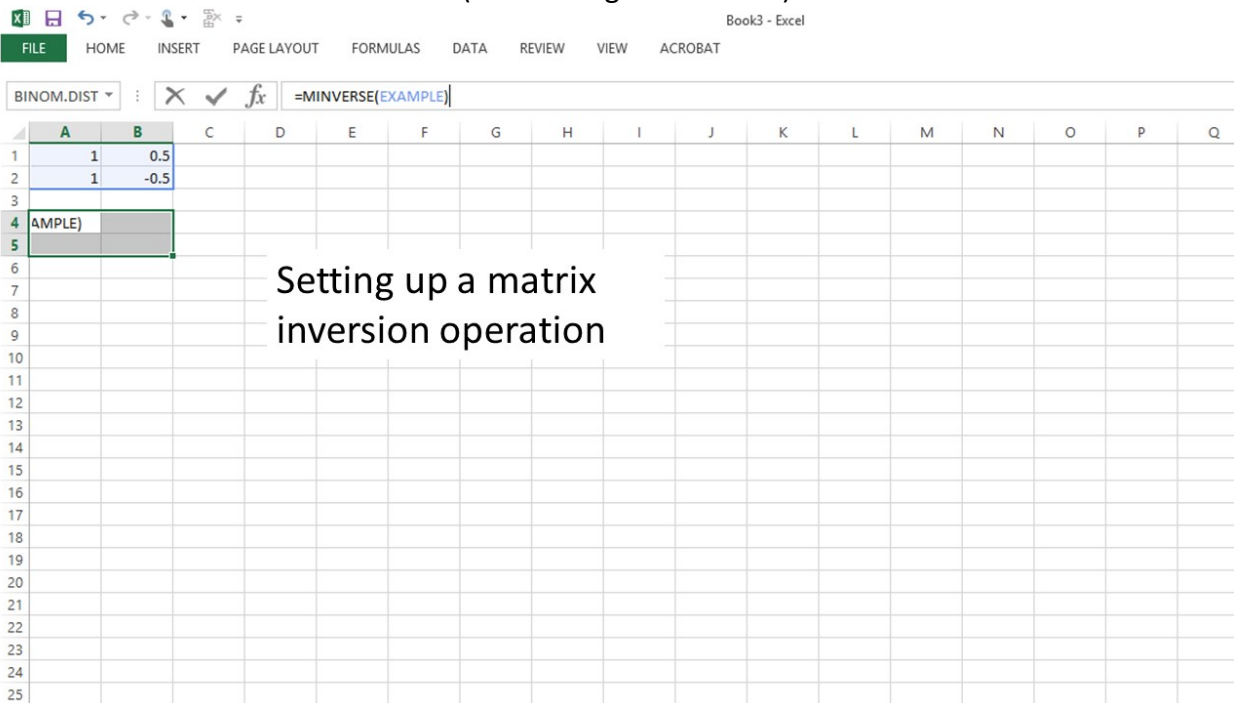
$$P = \begin{bmatrix} 1 & 0.5 \\ 1 & -0.5 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

To enter P, we can begin by entering the name in cell A1 and the elements in the block of cells A1:B2. To name the block of cells as matrix P, we do the following:

- (a) Highlight A1:B2 with the mouse.
- (b) Write the name of the matrix in the cell definition box. This is shown in the illustration below. The cell definition is in the red rectangle. In this case we can call the matrix EXAMPLE. We could have called it P or Q. However, some letters and even some combinations are reserved in Excel. Sometimes you will get errors if the name you choose conflicts. If you suspect such an error just choose another name.



We can demonstrate the use of matrix commands in Excel using this matrix. In the figure below we have set up the matrix inversion. The command is MINVERSE() and the argument is the name of the matrix. Notice that Excel highlighted the matrix in blue because it recognized EXAMPLE as the name of the matrix (as we designated above).



To execute the command and invert the matrix we need to type <Shift><ctrl><Enter>.

This is shown below. We see the result is a different matrix.

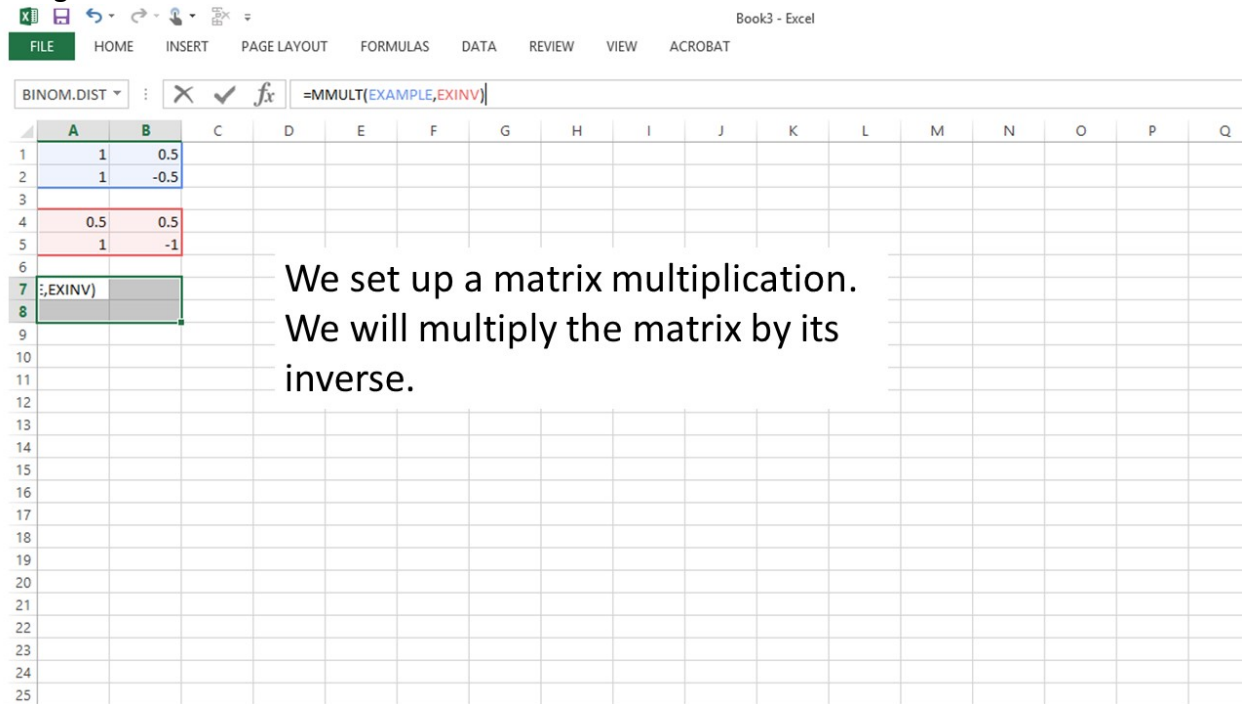
The screenshot shows the Excel interface with the formula bar containing `{=MINVERSE(EXAMPLE)}`. The spreadsheet has columns A through P and rows 1 through 23. A 2x2 matrix is defined in cells A1:B2 with values: A1=1, B1=0.5, A2=1, B2=-0.5. The result of the MINVERSE function is shown in cells A4:B5 with values: A4=0.5, B4=0.5, A5=1, B5=-1. A text box in the center of the grid reads: "To execute type <Shift><ctrl><Enter>".

First we will name it EXINV to represent that it is the inverse of the matrix EXAMPLE.

The screenshot shows the Excel interface with the Name Box set to "EXINV". The formula bar still contains `{=MINVERSE(EXAMPLE)}`. The spreadsheet layout is identical to the previous image, with the 2x2 matrix in A1:B2 and its inverse in A4:B5. A text box in the center of the grid reads: "You can name the inverse matrix in the same way you named the original matrix."

Next we will test this matrix to see if it really is the inverse matrix. We will use the matrix multiplication function to multiply the matrix EXAMPLE by its inverse. Below we can see that we

have set up the calculation and both matrices are highlighted, which means that Excel has recognized both of their names.

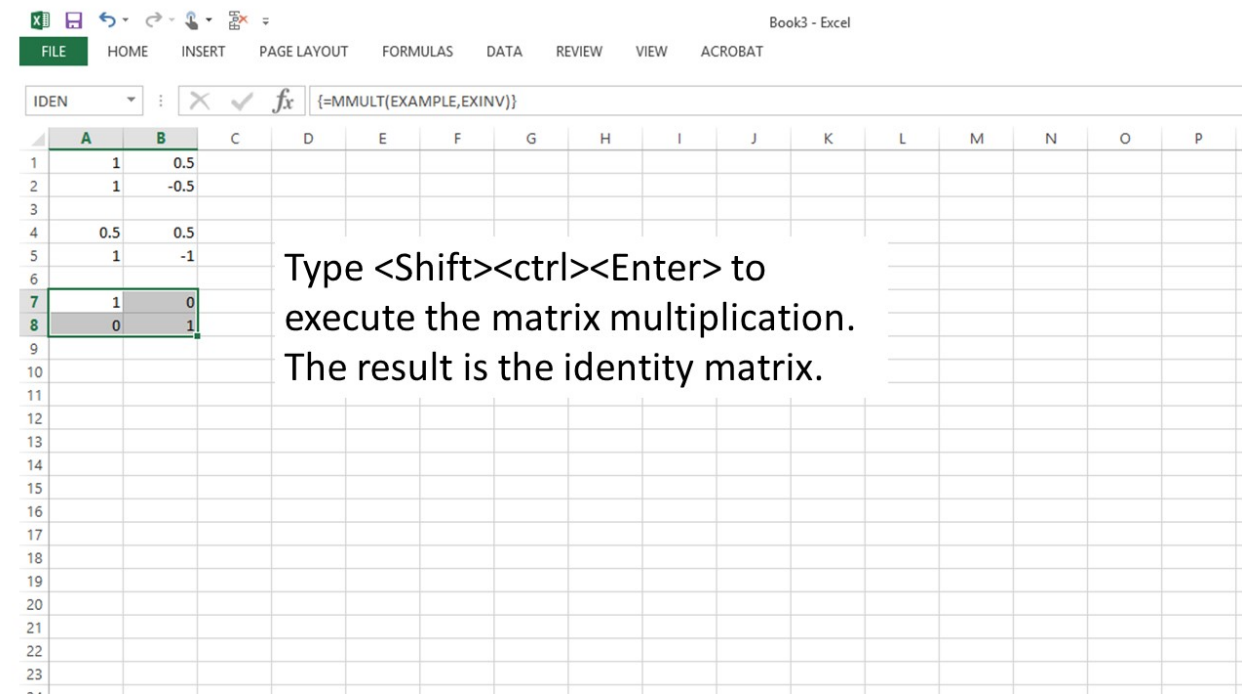


The screenshot shows the Excel interface with the formula bar containing `=MMULT(EXAMPLE,EXINV)`. The spreadsheet has two matrices highlighted: a 2x2 matrix in cells A1:B2 (values 1, 0.5, 1, -0.5) and a 2x2 matrix in cells A4:B5 (values 0.5, 0.5, 1, -1). The formula bar also shows `EXINV` in the adjacent cell.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	1	0.5															
2	1	-0.5															
3																	
4	0.5	0.5															
5	1	-1															
6																	
7	EXINV																
8																	
9																	
10																	
11																	
12																	
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We set up a matrix multiplication.  
We will multiply the matrix by its  
inverse.

Now we can execute the matrix multiplication by typing `<Shift><ctrl><Enter>`.



The screenshot shows the Excel interface with the formula bar containing `{=MMULT(EXAMPLE,EXINV)}`. The spreadsheet has two matrices highlighted: a 2x2 matrix in cells A1:B2 (values 1, 0.5, 1, -0.5) and a 2x2 matrix in cells A4:B5 (values 0.5, 0.5, 1, -1). The formula bar also shows `EXINV` in the adjacent cell.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	1	0.5														
2	1	-0.5														
3																
4	0.5	0.5														
5	1	-1														
6																
7	1	0														
8	0	1														
9																
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Type `<Shift><ctrl><Enter>` to  
execute the matrix multiplication.  
The result is the identity matrix.





In order to play around with the matrix operations, enter the two matrices above (P and Q). You can enter them anywhere you like. To keep things clear we should probably keep some spaces between them. So, for example we could enter the P matrix in A1:B2 (as above) and then the Q matrix in A4:B5. Please do this and then use the information below to implement various matrix operations.

## Matrix Operations

We need to know the following rules on matrix operations:

*P + Q or P - Q can be performed if P and Q are of the same size.*

*The multiplication PQ can be performed if the number of columns in P is the same as the number of rows in Q.*

*The **transpose** of P can always be performed.*

*The **determinant** of P is defined only if P is a square matrix.*

*The **inverse** of P is defined only if P is a square matrix with nonzero determinant.*

As the matrices P and Q are of the same size, we know P+Q can be performed. To do this in Excel, we use the following steps.

(a) Enter P and Q (which have been done already).

In the cells above those which will contain the answer we enter a description and the name of this answer. For example, we can enter **sum of P and Q** as the description, and enter RS as the name.

(b) Use the mouse to highlight a 2 x 2 block of cells (as P +Q will be a 2 x 2 matrix) such as B9:C10 in which the answer is to be stored.

(c) Enter the Excel function

$$= P + Q$$

followed by pressing <Shift><ctrl><Enter>

(d) Name the new matrix RS as we did above for the example.

**Note:** *You may like to name P + Q as R, but you will find this is not allowed as R is reserved in Excel for other use.*

For the other operations, you follow the same steps with the appropriate changes. The Excel function for

A-B is	= A - B
AB is	= MMULT(A,B)
$A^T$ (Transpose of A) is	= TRANSPOSE(A)
$A^{-1}$ (Inverse of A) is	= MINVERSE(A)
det(A) (Determinant of A) is	= MDETERM(A)

Note that you need to know the size of the resulting matrix in Step (c) before you perform the operation in Step (d). Remember, det(A) is a number, not a matrix (or, one can say it is a 1 x 1 matrix).

Task: Find  $P - Q$ ,  $PQ$ ,  $P^T$ ,  $Q^{-1}$ , det(P) and  $PT$  where

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

### Homework: Computer Lab 3

Answer the following questions to verify your progress.

1. What does the function =IMREAL(X) do?

A. It calculates the product of the imaginary and the real parts of X

B. The syntax is really =IMREAL(a,b) and it produces a complex number with a as imaginary and b as real part

C. It renders the real part of the complex number X

D. It swaps the real and imaginary parts of X

2. What are the four roots of the equation  $x^4=1$ ?

(Write out your answer in the space provided)

3. In step 12 you calculated the root of  $x^{15}=1$ . You could consider each root as a vector originating from the origin. The set of solutions then looks like the spokes of a bicycle wheel. What is the vector sum of all solutions? (Integer please)

Vector sum = \_\_\_\_\_.

4. In last weeks computer lab we have seen the hyperbolic 'trumpets' around a calibration line. Make a sketch showing how you would determine the errors in the value of an unknown using the line with trumpets.

### **Excel Spreadsheet Assignment**

An analyst determines the absorbance of a solution known to contain 4 organic compounds X=A, B, C and D at four different wavelengths  $\lambda= 435, 472, 513$  and  $570$  nm.

The extinction coefficients  $\epsilon_{\lambda}(X)$  for the 4 compounds at these 4 wavelengths are known (units lit/mol):

	A	B	C	D	
435	325	3.5	0.1	100	
472	50	200	590	0.1	
513	1290	700	4.3	12	
570	2	0.1	24	1350	

The values she finds for the absorbance in a cuvette with  $L=1\text{cm}$  are:

435	0.070259
472	0.480401
513	0.791769
570	0.433884

What are the four concentrations?

(Hint: Absorbance  $A = \epsilon_{\lambda}Lc$  is an additive quantity, so you can write out the problem as a set of linear equations. Then write this as a matrix formula and see if you can solve it by matrix algebra.

Submit the concentrations in webassign.

Hint: the equations have the form

$$A_1 = \epsilon_{11}c_1 + \epsilon_{12}c_2 + \epsilon_{13}c_3 + \epsilon_{14}c_4$$

$$A_2 = \epsilon_{21}c_1 + \epsilon_{22}c_2 + \epsilon_{23}c_3 + \epsilon_{24}c_4$$

$$A_3 = \epsilon_{31}c_1 + \epsilon_{32}c_2 + \epsilon_{33}c_3 + \epsilon_{34}c_4$$

$$A_4 = \epsilon_{41}c_1 + \epsilon_{42}c_2 + \epsilon_{43}c_3 + \epsilon_{44}c_4$$

We can write these compactly in matrix form as:

$$A = \epsilon c$$

Where the knowns are the vector  $A$  of absorbances and the matrix of the extinction coefficients. We can solve for the concentrations using the matrix inverse:

$$\epsilon^{-1}A = \epsilon^{-1}\epsilon c$$

Which tells us that

$$c = \epsilon^{-1}A$$

