Postulates of quantum mechanics

Quantum mechanics is a branch of science. Why does it need postulates? This name is a bit unusual. The issue is that we cannot prove the statements in a mathematical sense. Instead, these statements are self-consistent rules that apply to quantum mechanical equations and systems.

Postulate 1. Any state of a system of N particles can be described by a wave function Ψ.

$$\Psi = \Psi(1,2,3,\ldots N,t)$$

Corollary 1.1: the probability density of the particles in the system is defined as $\Psi^*\Psi$. The wave function, Ψ, must be continuous and differentiable. The function must single valued. Finally, the square of the wave function, defined as $\Psi^*\Psi$, must integrable. These mathematical requirements are needed to ensure that the wave function is smooth and lacks discontinuities that would be physically unreasonable. The idea that the probability density must integrable arises since the probability is calculated by

$$P_{ab} = \int_a^b \Psi^*\Psi d\tau$$

where the differential volume element, $d\tau$, is relevant to the space being considered.

Corollary 1.2: A meaningful probability in quantum mechanics can only be calculated if the wave function is properly normalized. The normalization condition states that the probability of finding the particle somewhere in the relevant space is equal 1.

$$\int_{-\infty}^{\infty} \Psi^{*}\Psi d\tau = 1$$

The limits of the integral need to be set according to the system so that the encompass all of the space that the particle/wave may occupy. We will show a practical method to normalize a wave function so that this condition can always be met.

Postulate 2. Every observable property of a system has a corresponding linear Hermitian operator. Since wave functions can be complex and physical properties are calculated by an operator equation, it is important that the operator always give a real eigenvalue. The value of the energy, momentum etc. cannot be complex. The Hermitian property of the operator guarantees that the observable will be real.

Postulate 3. If the operator $\hat{\alpha}$ corresponds to an observable for a set of identical systems in s state and ψ is an eigenfunction of $\hat{\alpha}$ with eigenvalue *a*, such that,

$$\hat{\alpha}\psi = a\psi$$

then, a series of measurements on different members of the set always leads to the value *a*.

Postulate 4. If the operator $\hat{\beta}$ corresponds to an observable for a set of identical systems in a state and ψ_i is an eigenfunction of $\hat{\beta}$ then the average of a series of measurements on different members of the set is given by

$$\langle b \rangle = \frac{\langle \psi_i | \hat{\beta} | \psi_i \rangle}{\langle \psi_i | \psi_i \rangle}$$

The quantity $\langle b \rangle$ is the average value of β, and it is also known as the expectation value.

Postulate 5. The state function $\psi(t)$ evolves with time as

$$H\psi(t) = i\hbar \frac{\partial}{\partial t}\psi(t)$$

The time dependent wave function compatible with this definition is:

$$\psi(t) = \psi e^{-iEt/\hbar}$$

where the stationary wave function, ψ, is a solution of the equation

$$H\psi = E\psi$$

Definition of Commutator

The quantum mechanical momentum operator can be derived from the energy operator. Since

$$\frac{p^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

We can conclude that

$$p = -i\hbar\frac{\partial}{\partial x}$$

Postulate 6: the eigenvalues of two operators will be simultaneously measurable to any accuracy if those operators commute. We define the commutator as:

$$[a, b] = ab - ba$$

Definition of a test function

In order to determine whether two operators commute we usually need to operate on a test function, which is an eigenfunction of the operators. For example, the wave function,

$$\psi(x) = e^{-ikx}$$

$$pe^{-ikx} = i\hbar\frac{\partial}{\partial x}e^{-ikx} = \hbar k e^{-ikx}$$

is an eigenfunction of the momentum. We can see this since

There is also a test function for the time-dependent operator introduced in posulate 5,

$$\psi(t) = e^{-i\omega t}$$

such that

$$He^{-i\omega t} = i\hbar\frac{\partial}{\partial t}e^{-i\omega t} = \hbar\omega e^{-i\omega t}$$

H and p commute

We now ask whether the energy and momentum are simultaneously measurable. If the operators commute, then we can say that energy and momentum can be measured simultaneously to arbitrary accuracy (i.e. to the best possible accuracy under experimental conditions).

$$[H,p]\psi(x) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x) - \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x)$$

If we factor out the constants we find that

$$\frac{i\hbar^3}{2m}\frac{\partial^3}{\partial x^3}\psi(x) - \left(\frac{i\hbar^3}{2m}\frac{\partial^3}{\partial x^3}\right)\psi(x) = 0$$

From this reasoning we see that the momentum and energy are simultaneously measurable in a quantum mechanical system.

p and x do not commute

Now we consider momentum, p, and position, x. The commutator is,

$$[p,x]e^{ikx} = \left(-i\hbar\frac{\partial}{\partial x}\right)xe^{ikx} - x\left(-i\hbar\frac{\partial}{\partial x}\right)e^{ikx}$$

In this case we see that we must apply the product rule to the first term,

$$\left(-i\hbar\frac{\partial}{\partial x}\right)xe^{ikx} = -i\hbar e^{ikx} - \hbar kxe^{ikx}$$

The second term is

$$-x\left(-i\hbar\frac{\partial}{\partial x}\right)e^{ikx} = \hbar kxe^{ikx}$$

Two of the terms cancel and we have

$$[p,x] = -i\hbar$$

E and t do not commute

A similar relationship holds for energy and time. To see this we define the time-dependent energy Hamiltonian as using the equation

$$-i\hbar\frac{\partial}{\partial t}\Psi(t) = E\Psi(t)$$

One solution to this equation is

$$\Psi(t) = e^{iEt/\hbar}$$

The mathematical solution of the time-energy commutator is entirely analogous to the position-momentum commutator.

$$[E,t]e^{\frac{iEt}{\hbar}} = \left(-i\hbar\frac{\partial}{\partial t}\right)te^{iEt/\hbar} - t\left(-i\hbar\frac{\partial}{\partial t}\right)e^{iEt/\hbar}$$

Once again two of the terms cancel and we have

$$[E,t] = -i\hbar$$

Significance of commutator for [p,x] and [E,t]

The momentum and position do not commute. This is also a statement of the Uncertainty Principle. The position and momentum of particle are not simultaneously measurable with arbitrary accuracy. Instead, there is a limitation on how accurately we can measure both the position and momentum simultaneously.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

A similar comment holds for energy and time. We can see the Precise (mathematical) analogy in the form of the commutators for these two quantities. They both have the same form.

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$